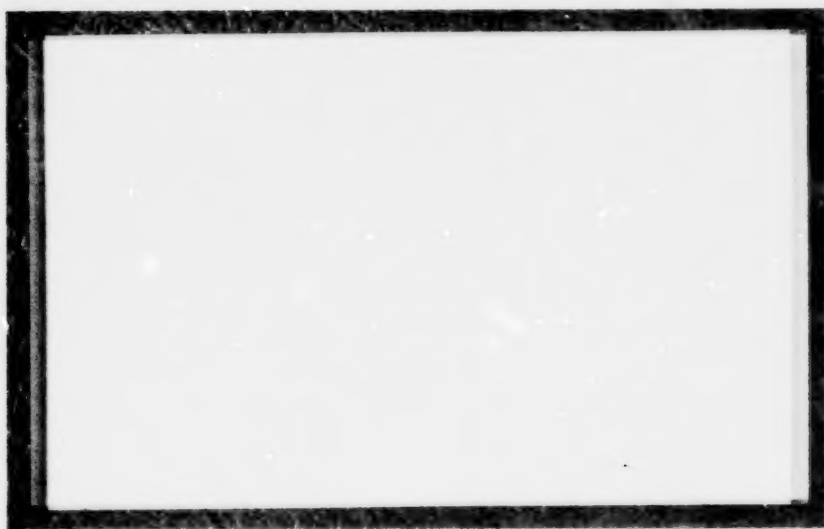


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CENTRE DE RECHERCHE SUR LES TRANSPORTS



A NOTE ON MAXIMAL MULTICOMMODITY
NETWORK FLOWS

by

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ABSTRACT

In this note, we propose an algorithm for the solution of the maximum multicommodity flow problem in a capacitated network. This algorithm requires the solution of a sequence of minimum convex cost multicommodity flow problems, which have the same structure as that of the network equilibrium problem with variable demand.

Key words : maximum multicommodity flow problem, augmented Lagrangean, network equilibrium problem with variable demand.

RÉSUMÉ

Nous proposons une procédure de solution pour le problème du flot maximal à plusieurs produits sur un réseau avec capacités sur les arcs, en utilisant une reformulation du problème basée sur l'approche du Lagrangean augmenté. Cette procédure demande la solution d'une suite de problèmes de multiflots à coût connexe minimal, qui ont la même structure que le problème d'équilibre de réseau avec demande variable.

Mots-clés : flot maximal à plusieurs produits, Lagrangean augmenté, problème d'équilibre de réseau avec demande variable.

1. Introduction

The purpose of this note is to propose an algorithm for the solution of the maximum multicommodity network flow problem in a capacitated network, based on an Augmented Lagrangean reformulation of the problem. As we shall show, this way of considering this problem requires the solution of a sequence of minimum convex cost multicommodity flow problems which have the same form as a network equilibrium problem with variable demand, for which algorithms have been developed that can solve large scale problems efficiently. Previous approaches to solve this problem resort to price directive decomposition methods for large scale linear programs (Ford and Fulkerson (1958)) or resource directive decomposition methods for large scale linear programs (Grinold (1968), Held et al (1974), Kennington and Shakley (1976)). A heuristic method for solving the problem has been suggested by Gratzner and Steiglitz (1972) based on notions derived from a resource directive decomposition method. Exact methods for solving this problem are known only for the case of two commodity maximal flow (Rothschild and Whinston (1966) and Sakarovitch (1973)).

2. The Problem

We consider a network (N, A) where N is the set of nodes and A is the set of directed arcs. The nodes $n, n \in N$, represent origins, destinations and intersections of links; the directed arcs $a, a \in A$, represent the transportation facilities. The flows on link a is v_a and the unknown maximal flows for origin/destination pairs $i, i \in I \subset N \times N$, are g_i . The origin to destination flows g_i may use directed paths $k, k \in K_i$, and the path flows are denoted h_k .

The maximum multicommodity network flow problem in a capacitated network, where each arc is subject to a capacity constraint c_a , may be formulated as

$$\text{Max } \sum_{i \in I} g_i \quad (1)$$

$$\text{Subject to } \sum_{k \in K_i} h_k - g_i = 0, i \in I \quad (2)$$

$$h_k \geq 0, k \in K_i, i \in I \quad (3)$$

$$v_a \leq c_a, a \in A \quad (4)$$

$$g_i \geq 0, i \in I \quad (5)$$

$$v_a = \sum_{i \in I} \sum_{k \in K_i} \delta_{ak} \cdot h_k, a \in A \quad (6)$$

$$\text{where } \delta_{ak} = \begin{cases} 1 & \text{if link } a \text{ belongs to path } k \\ 0 & \text{otherwise} \end{cases}$$

The problem has been formulated in the space of path flows in order to simplify the statement of the conservation of flow equations, as in (2). The nonnegativity constraints on the path flows (3) ensure that $v_a \geq 0, a \in A$.

In this formulation, a distinct commodity is associated with each distinct origin/destination pair. If two or more commodities are associated with some origin/destination pairs, then an additional index p would be needed to denote the commodities, e.g. g_i^p .

3. Solution by the Augmented Lagrangean Method

The reformulation of the problem (1) - (5) for solution by the Augmented Lagrangean method consists in dualizing constraints (4), which results in the following problem (see Bertsekas (1982))

$$\text{Min}_{v,g} \text{Max}_{\lambda \geq 0} L_u(v,g,\lambda) = -\sum_{i \in I} g_i + \frac{1}{2U} \sum_{a \in A} \left\{ \max(0; \lambda_a + U(v_a - c_a))^2 - \lambda_a^2 \right\} \quad (7)$$

Subject to (2), (3), (5), (6)

where U is the penalty constant and $\lambda_a, a \in A$, are the Lagrange multipliers associated with each constraint in (4).

The solution of this problem is obtained by the following general algorithm:

Step 0 (Initialization)

$l := 0$; $\lambda_{a,l} := 0, a \in A$ where l is an iteration index

$U_l := \bar{U}$, where $\bar{U}$ is an initial value of the penalty constant

Step 1 (Solve problem in v, g for fixed λ)

$$\underset{v, g}{Min} \quad - \sum_{i \in I} g_i + \frac{1}{2U_l} \sum_{a \in A} \left\{ \max(0; \lambda_{a,l} + U_l (v_a - c_a))^2 - \lambda_{a,l}^2 \right\} \quad (8)$$

Subject to (2), (3), (5), (6)

Let $(v_a, i; g_{i,l})$ be the solution of this problem

Step 2 (Adjustment of multipliers)

$$\lambda_{a, l+1} := \max(0; \lambda_{a, l} + U_l (v_a - c_a)), \quad a \in A$$

Step 3 (Optimality Test)

If $\|\lambda_l - \lambda_{l+1}\| \leq \varepsilon$, STOP.

Otherwise continue to

Step 4 (Adjustment of penalty constant)

$$U_{l+1} \geq U_l$$

$l := l+1$ and return to Step 1.

The subproblem to be solved in Step 1 is a convex cost multicommodity network flow problem which appears at first to be difficult, however a closer inspection reveals that it has the same structure as a network equilibrium problem with variable demand. In order to do so we recall its formulation.

The network equilibrium problem with variable demand seeks to compute the origin to destination demands g_i which are given by monotone decreasing demand functions $G_i(u_i)$, $i \in I$, where u_i is the least cost of travel for origin/destination pair i , and the arc flows v_a , $a \in A$, where the cost of traveling on a link $s_a(v_a)$, $a \in A$, is a monotone increasing function. This problem, as is well known, is formulated as:

$$\text{Min} - \sum_{i \in I} \int_0^{g_i} W_i(y) dy + \sum_{a \in A} \int_0^{v_a} s_a(x) dx \quad (9)$$

Subject to (2), (3), (5), (6)

where $W_i(g_i)$ is the inverse of the demand function $G_i(u_i)$.

If we juxtapose the corresponding terms of (8) and (9) it is possible to observe that $-\sum_{i \in I} g_i$ may be rewritten as $-\sum_{i \in I} \int_0^{g_i} dy$ and that $\sum_{a \in A} \left\{ \max(0; \lambda_{a,l} + U_l(v_a - c_a))^2 - \lambda_{a,l}^2 \right\}$ is a convex function of v_a which may be stated as $\sum_{a \in A} S_a(v_a)$, where $S_a(v_a)$ is defined to be the expression in the curly brackets. Thus (7) may be rewritten as

$$\text{Min}_{v, \lambda} - \sum_{i \in I} \int_0^{g_i} dy + \sum_{a \in A} S_a(v_a) \quad (10)$$

and the similarity between the objectives (8) and (9) are evident as $\tilde{S}_a(v_a) = \int_0^{v_a} s_a(x)dx$ is a convex function.

The purpose of this exercise is to conclude that (8) is a rather easy problem to solve for which a plethora of first order solution algorithms have been developed (see Florian (1986)), the most common being the linear approximation algorithm of Frank and Wolfe (1956) and its variants. It is important to note that (8) may not be solved by second order methods, since the objective function is only once differentiable. Furthermore, relatively large scale problems (~ 2 000-4 000 nodes, 6 000 - 12 000 arcs) may be solved in reasonable computational times. It is interesting to note, that given this observation, (8) has the structure of a network equilibrium problem where the inverse demand functions are constant and equal to 1.

4. Some Comments

The approach suggested in this note is suitable for computing maximum multicommodity flows for large networks where the arc capacities do not necessarily represent strict constraints.

From the computational viewpoint, if a primal algorithm is used in Step 1 to solve (7), the solution from iteration l is feasible for iteration $l+1$ since only the objective function is modified in steps 2 and 4. Thus, at iteration $l+1$ only a reoptimization of (8) is required where $(v_a, i, g_{i,l})$ serve as an initial feasible solution. In an eventual implementation of this algorithm, the particular choices made for values of the penalty constant U and other possible stopping criteria that may be used, would need careful attention.

It is also interesting to draw some conclusions regarding the uniqueness of the solution from this way of reformulating the maximum multicommodity flow problem. While it

is known that the solution of a linear program is not necessarily unique, by inspecting (10) it is easy to draw stronger immediate conclusions. Since the coefficient of each variable g_i , $i \in I$ is constant, it follows that the demands g_i are not unique; in fact the only variables which are unique are the arc flows v_a which equal to the capacity c_a in an optimal solution. The consequence is that this model is best used to find the "bottleneck" arcs in a multicommodity network and care must be exercised to interpret the optimal values of the demands g_i , $i \in I$.

This approach to solving for maximal multicommodity flows may be easily extended to consider lower bounds on the demands g_i , $i \in I$ and "weights" $\gamma_i, \gamma_i > 0$, which would give preference to one product over others (i.e. $\text{Max} \sum_{i \in I} \gamma_i g_i$).

Finally, it is sometimes revealing to inspect the Kuhn-Tucker conditions of a convex optimization problem subject to linear constraints. These are, for (10) subject to (2), (3), (5), (6) :

$$\left. \begin{aligned} \sum_{a \in A} \delta_{ak} S'_a(v_a) - u_i - \gamma_k &= 0 \\ h_k \cdot \gamma_k &= 0 \\ \gamma_k &\geq 0 \end{aligned} \right\} k \in k_i, i \in I \quad (11)$$

$$\text{and} \quad -1 + u_i = 0 \quad i \in I \quad (12)$$

where u_i, γ_k are the dual variables corresponding to (2), (3) respectively. For each path h_k such that $h_k > 0$, it follows, by using (11) and (12), that the "length" of the path, $\sum_{a \in A} \delta_{ak} S'_a(v_a) = 1$ whatever the values of U and λ_a , $a \in A$. This is interesting, but of little practical use.

5. Conclusion

We have showed in this note that the maximum multicommodity flow problem may be solved by a sequence of convex cost multicommodity flow problems, by using an Augmented Lagrangean reformulation of the problem. In particular the convex cost problems that must be solved have the same structure as the network equilibrium problem with variable demand, which is a relatively easy problem to solve. Thus, one may solve for maximum multicommodity flows in a capacited network without having to resort to primal or dual decomposition methods of linear programming. In the near future, we will report on computational results obtained with this algorithmic approach on applications which relate to the transportation of freight over transportation networks.

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